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MATHEMATICS
TARGET IIT JEE

APPLICATION OF DERIVATIVE

(TANGENT & NORMAL, MONOTONOCITY, MAXIMA & MINIMA)

THEORY AND EXERCISE BOOKLET

CONTENTS

TANGENT & NORMAL

MONOTONOCITY

MAXIMA & MINIMA

ANSWER KEY

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JEE Syllabus :

Tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, applications of Rolle's Theorem and Lagrange's Mean Value Theorem.

TANGENT & NORMAL

A. TANGENT & NORMAL

Definition : The tangent line to the graph of f at the point P(a, f(a)) is

(1) the line on P with slope $f'(a)$ if $f'(a)$ exists ;

(2) the line x = a if $\lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}\right| = \infty$.

In neither (1) nor (2) holds, then the graph of does not have a tangent line at the point P(a, f(a)).

In case $f'(a)$ exists, then $y - f(a) = f'(a)(x - a)$

is an equation of the tangent line to the graph of f at the point P(a, f(a)).

The normal line N to the graph of a function f at the point P(a, f(a)) is defined to be the line through P perpendicular to the tangent line.

It follows that if $f'(a) \neq 0$ the slope of N is $-1/f'(a)$ and

$y - f(a) = -\frac{1}{f'(a)}(x - a)$ is an equation of N.

If $f'(a) = 0$, then N is the vertical line x = a; and if the tangent line is vertical, then N is the horizontal line y = f(a).

Note :

1. The point $P(x_1, y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.

2. If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

3. If the tangent at any point on the curve is parallel to the axis of y, then $dy/dx = \infty$ or dx/dy = 0.

4. If the tangent at any point on the curve is equally inclined to both the axes there $dy/dx = \pm 1$.

5. For equation of tangent at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x+x_1}{2}$ for x, $\frac{y+y_1}{2}$ for y and $\frac{xy_1+x_1y}{2}$ for xy and keep the constant as such. This method is applicable only for second degree curves, i.e., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

6. Method to find normal at (x_1, y_1) of second degree conics $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The equation of normal at (x_1, y_1) is $\frac{x-x_1}{ax_1+hy_1+g} = \frac{y-y_1}{hx_1+by_1+f}$

Ex.1 Find the equation of tangent to the ellipse $3x^2+y^2+x+2y = 0$ which are perpendicular to the line 4x – 2y = 1.

Sol. Since, tangent is the perpendicular to the line 4x – 2y = 1,

$\therefore$ (slope of tangent) × (slope of normal) = –1 $\Rightarrow \frac{dy}{dx} \times 2 = -1 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$ ....(i)

The given equation $3x^2 + y^2 + x + 2y = 0$ ....(ii)

$\Rightarrow 6x + 2y\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{(6x+1)}{2(y+1)}$ ....(iii)

Let (x_1, y_1) be the point of contact of the tangent and the curve

From (i) and (iii), we get $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{(6x_1+1)}{2(y_1+1)} = -\frac{1}{2}$ i.e., $y_1 = 6x_1$ ....(iv)

Substituting this in (ii) [since the points lies on the curve] we get,

$3x_1^2 + 36x_1^2 + x_1 + 12x_1 = 0$ i.e., $13x_1(3x_1 + 1) = 0$ $\Rightarrow$ $x_1 = 0, -1/3$

Using (iv), $x_1 = 0$ $\Rightarrow$ $y_1 = 0$ and $x_1 = -1/3$ $\Rightarrow$ $y_1 = -2$

Hence, the points where tangent has slope −1/2 are P(0, 0) and Q(−1/3, −2).

Equation of tangents at P, Q are $y = -\frac{1}{2}x$ i.e., $x + 2y = 0$

and $y + 2 = -\frac{1}{2}\left(x + \frac{1}{3}\right)$ i.e., $3x + 6y + 13 = 0$ respectively.

Ex.2 Find the equation of normal to the curve $x + y = x^y$, where it cuts x–axis.

Sol. Given curve is $x + y = x^y$ .....(i) at x–axis y = 0, $\therefore$ $x + 0 = x^0$ $\Rightarrow$ $x = 1$

$\therefore$ Point is A(1, 0)

Now to differentiation $x + y = x^y$ taking log of both sides

$\Rightarrow$ $\log(x + y) = y \log x$ $\therefore$ $\frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y \cdot \frac{1}{x} + (\log x)\frac{dy}{dx}$

Putting x = 1, y = 0 $\left\{1+\frac{dy}{dx}\right\} = 0$ $\Rightarrow$ $\left(\frac{dy}{dx}\right)_{(1,0)} = -1$ $\therefore$ slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1} = 1$ $\Rightarrow$ $y = x - 1$

Ex.3 At what points on the curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, the tangents make equal angles with co-ordintae axes ?

Sol. Given curve is $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$ ...(1)

Differentiating both sides w.r.t.x, then $\frac{dy}{dx} = 2x^2 + x$

$\because$ Tangent make equal angles with co-ordinate axes $\therefore$ $\frac{dy}{dx} = \pm 1$ **or** $2x^2 + x = \pm 1$

or $2x^2 + x + 1 \neq 0$ **and** $2x^2 + x - 1 = 0$

or $2x^2 + 2x - x - 1 = 0$ **or** $(2x - 1)(x + 1)$ $\therefore$ $x = \frac{1}{2}, -1$ **(If** $2x^2 + x + 1 = 0$ **then x is imaginary)**

From (1), for $x = \frac{1}{2}$, $y = \frac{2}{3}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{4} = \frac{5}{24}$ **and for** $x = -1$, $y = -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}$

Hence points are $\left(\frac{1}{2}, \frac{5}{24}\right)$ **and** $\left(-1, \frac{1}{6}\right)$.

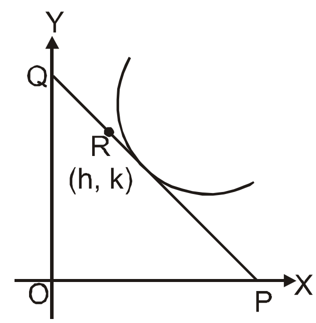
Ex.4 **The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the mid-point of PQ is a circle.**

Sol. The given curve is $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$

$\therefore$ Equation of tangent at 'θ' is $Y - a\sin^3\theta = -\tan\theta\,(X - a\cos^3\theta)$

$\Rightarrow \frac{Y}{\sin\theta} - a\sin^2\theta = -\frac{X}{\cos\theta} + a\cos^2\theta$

$\Rightarrow \frac{X}{\cos\theta} + \frac{Y}{\sin\theta} = a$ or $\frac{X}{(a\cos\theta)} + \frac{Y}{(a\sin\theta)} = 1$



$\therefore$ $P \equiv (a\cos\theta, 0)$ and $Q \equiv (0, a\sin\theta)$

If mid-point of PQ is R(h, k), then $2h = a\cos\theta$ and $2k = a\sin\theta$

$\therefore$ $(2h)^2 + (2k)^2 = a^2$ or $h^2 + k^2 = a^2/4$

Hence locus of mid-point is $x^2 + y^2 = a^2/4$ which is a circle.

Ex.5 Show that the curve $x = 1 - 3t^2$, $y = t - 3t^3$ is symmetrical about x-axis and has no real point for $x > 1$. If the tangent at the point t is inclined at an angle ϕ to OX. Prove that $3t = \tan\phi + \sec\phi$. If the tangent at P(−2, 2) meets the curve again at Q, prove that the tangents at P and Q are at right angles.

Sol. Given curve is $x = 1 - 3t^2$ ...(1) & $y = t - 3t^3$ ...(2)

From (1) and (2), $y = tx$ or $x = 1 - 3\left(\frac{y}{x}\right)^2$ $\Rightarrow$ $x^3 = x^2 - 3y^2$

Since all powers of y are even, so curve is symmetrical about x-axis.

For $x > 1$ $\Rightarrow$ $1 - 3t^2 > 1 \Rightarrow$ $-3t^2 > 0$ Impossible

From (1) and (2), $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1-9t^2}{-6t} = \tan\phi$ (given) ...(3)

$\therefore$ $\sec^2\phi = 1 + \tan^2\phi = 1 + \left(\frac{1-9t^2}{6t}\right)^2 = \left(\frac{1+9t^2}{6t}\right)^2$ $\therefore$ $\sec\phi = \frac{1-9t^2}{6t}$

Adding (3) and (4) we get, $\tan\phi + \sec\phi = 3t$

$\because$ P(−2, 2) $\therefore$ $1 - 3t^2 = -2$ and $2 = t - 3t^3$ then we get $t = -1$

$\therefore$ $\left.\frac{dy}{dx}\right|_{t=-1} = \frac{1-9(-1)^2}{-6(-1)} = -\frac{4}{3}$

Equation of tangent at (−2, 2) is $Y - 2 = -\frac{4}{3}(X + 2)$

$\Rightarrow$ $t - 3t^3 - 2 = -\frac{4}{3}(1 - 3t^2 + 2)$ $\Rightarrow$ $3t - 9t^3 - 6 = -12 + 12t^2$

$\Rightarrow \quad 9t^3 + 12t^2 - 3t - 6 = 0 \qquad \Rightarrow \qquad (t+1)^2 (3t - 2) = 0$

Therefore the tangent at t = –1 meets the curve again at $t = \frac{2}{3}, Q\left(-\frac{1}{3}, -\frac{2}{9}\right)$

$\therefore \quad \left.\frac{dy}{dx}\right|_{t=2/3} = \frac{1-9\left(\frac{4}{9}\right)}{-6\left(\frac{2}{3}\right)} = \frac{3}{4}$ Hence $\quad \left.\frac{dy}{dx}\right|_{t=-1} \times \left.\frac{dy}{dx}\right|_{t=-2/3} = -1$

Hence the tangents at P and Q are at right angles.

Ex.6 Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Sol. Equation of tangent at (2, 8) is $y = 12x - 16$

Solving this with $y = x^3$ we get $x^3 - 12x + 16 = 0$

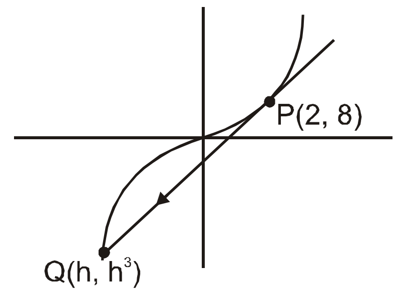
this cubic must give all points of intersection of line and curve $y = x^3$

i.e., point P and Q.

But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations

sum of roots $\quad \Rightarrow \quad 2 + 2 + h = 0 \quad \Rightarrow \quad h = -4$

Hence coordinates of Q are (–4, –64)



Ex.7 If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos\phi - x \sin\phi = a \cos 2\phi$.

Sol. Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(1)

Differentiating both sides w.r.t.x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \qquad \therefore \qquad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

Slope of normal = $-dx = \frac{dx}{dy} = \frac{x^{1/3}}{y^{1/3}} = \tan\phi$ (given) $\qquad \therefore \qquad x = y\tan^3\phi \qquad$ ...(2)

From (1) and (2), $y^{2/3}(1 + \tan^2\phi) = a^{2/3} \Rightarrow y^{2/3} = a^{2/3}\cos^2\phi \quad \therefore \quad y = a\cos^3\phi$ and $x = a\sin^3\phi$

Therefore equation of normal is $y - a\cos^3\phi = \tan\phi\,(x - a\sin^3\phi)$

$\Rightarrow \quad y\cos\phi - a\cos^4\phi = x\sin\phi - a\sin^4\phi$

$\Rightarrow \quad y\cos\phi - x\sin\phi = a(\cos^4\phi - \sin^4\phi) = a(\cos^2\phi + \sin^2\phi)(\cos^2\phi - \sin^2\phi) = a \cdot 1 \cdot \cos 2\phi$

Hence $y\cos\phi - x\sin\phi = a\cos 2\phi$.

Ex.8 **(a)** Find y′ if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what points on the curve is the tangent line horizontal ?

Sol. **(a)** Differentiating both sides $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the y^3 term and the Product Rule on the 6xy term, we get

$$3x^2 + 3y^2y' = 6y + 6xy' \qquad \text{or} \qquad x^2 + y^2y' = 2y + 2xy'$$

We now solve for y′ : $\qquad y^2y' - 2xy' = 2y - x^2$

$$(y^2 - 2x)y' = 2y - x^2, \qquad y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When x = y = 3, $y' = \dfrac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = -1$

So an equation of the tangent to the folium at (3, 3) is $y - 3 = -1(x - 3)$ or $x + y = 6$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$. Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get $x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$

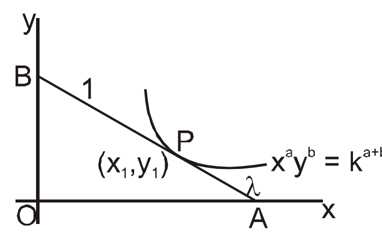
which simplifies to $x^6 = 16x^3$. so either $x = 0$ or $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$.

Thus, the tangent is horizontal at (0, 0) and at $(2^{4/3}, 2^{5/3})$.

Ex.9 In the curve $x^a y^b = k^{a+b}$, (a b > 0) prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in constant ratio.

Sol. Let $P(x_1, y_1)$ be the point of contact of the tangent.

Here, $x^a y^b = k^{a+b}$ $\therefore$ $a \log x + b \log y = (a + b) \log k$.



Differentiating, $\dfrac{a}{x} + \dfrac{b}{y}\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = -\dfrac{ay}{bx}$; $\therefore$ $\left(\dfrac{dy}{dx}\right)_{x_1, y_1} = \dfrac{-ay_1}{bx_1}$

$\therefore$ the equation of the tangent at $P(x_1, y_1)$ is

$y - y_1 = \left(\dfrac{dy}{dx}\right)_{x_1, y_1} . (x - x_1)$ or $y - y_1 = \dfrac{-ay_1}{bx_1}(x - x_1)$ ...(1)

Solving with y = 0, $-y_1 = \dfrac{-ay_1}{bx_1}(x - x_1)$ or $bx_1 = a(x - x_1)$; $\therefore$ $x = \dfrac{(a+b)x_1}{a}$

Solving (1) with x = 0, $y - y_1 = \dfrac{-ay_1}{bx_1}(-x_1)$ or $y = y_1 + \dfrac{ay_1}{b} = \dfrac{(a+b)y_1}{b}$

$\therefore$ $A = \left(\dfrac{a+b}{a}x_1, 0\right)$ and $B = \left(0, \dfrac{a+b}{b}y_1\right)$

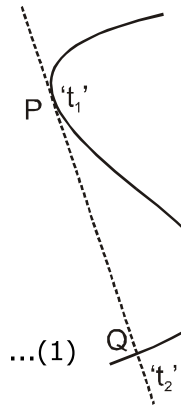
Let P divide AB in the ratio λ : 1. Then $P = \left(\dfrac{\lambda.0 + 1.\frac{a+b}{b}x_1}{\lambda+1}, \dfrac{\lambda.\frac{a+b}{b}y_1 + 1.0}{\lambda+1}\right) = \left(\dfrac{a+b}{a(\lambda+1)}x_1, \dfrac{\lambda(a+b)}{b(\lambda+1)}y_1\right)$

$\therefore$ $x_1 = \dfrac{a+b}{a(\lambda+1)}x_1$ and $\dfrac{\lambda(a+b)}{b(\lambda+1)}y_1$ $\Rightarrow$ $a(\lambda + 1) = a + b$ and $b(\lambda + 1) = \lambda(a + b)$

$\Rightarrow$ $\lambda = \dfrac{b}{a}$ and $b = \lambda a$, i.e., $\lambda = \dfrac{b}{a}$ $\therefore$ P divides AB in the constant ratio b : a.

Ex.10 Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.

Sol. Let the tangent to the curve at P't_1', i.e., $(4t_1^2 + 3,\ 8t_1^3 - 1)$ be normal to the curve at Q't_2', i.e., $(4t_2^2 + 3,\ 8t_2^3 - 1)$.



The equation of the tangent at 't_1' is $y - (8t_1^3 - 1) = \left(\frac{dy}{dx}\right)_{t_1} . \{x - (4t_1^2 + 3)\}$

or $\quad y - (8t_1^3 - 1) = \left(\frac{dy/dt}{dx/dt}\right)_{t_1} . \{x = (4t_{12} + 3)\}$

or $y - (8t_1^3 - 1) = \frac{24t_1^2}{8t_1} . \{x - (4t_1^2 + 3)\}$ or $y - (8t_1^3 - 1) = 3t_1\{x - (4t_1^2 + 3)\}$ ...(1)

Clearly, slope of the tangent at 't_1' = slope of the normal at 't_2'

and (1) passes through the pont 't_2'. $\quad \therefore \quad \left(\frac{dy}{dx}\right)_{t_1} = \frac{-1}{\left(\frac{dy}{dx}\right)_{t_2}}$, .i.e., $3t_1 = \frac{-1}{3t_2}$ ...(2)

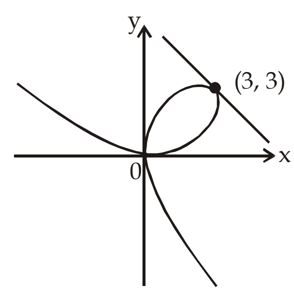
and $(8t_2^3 - 1) - (8t_1^3 - 1) = 3t_1\{(4t_2^2 + 3) + (4t_1^2 + 3)\}$

$\Rightarrow \quad 8(t_2^3 - t_1^3) = 3t_1 . 4(t_2^2 - t_1^2) \qquad \Rightarrow \quad 2(t_2^2 + t_2t_1 + t_1^2)\ 3t_1(t_2 + t_1)$

$\Rightarrow \quad 2t_2^2 + t_1t_2 + t_1^2$, i.e., $2\left(\frac{-1}{9t_1}\right)^2 = \frac{-1}{9} + t_1^2 \qquad$ {using (2)}

$\Rightarrow \quad 2 = -9t_1^2 + 81t_1^4 \qquad \therefore \quad 81t_1^4 - 9t_1^2 - 2 = 0 \quad$ or $\quad (9t_1^2 + 1)(9t_1^2 - 2) = 0$

$\therefore \quad 9t_1^2 - 2 = 0; \qquad \therefore \quad t_1 = \pm \frac{\sqrt{2}}{3}$



Figure

$\therefore$ Putting in (1), the equations of the required lines are

$$y - \left(8.\frac{\pm 2\sqrt{2}}{27} - 1\right) = 3\left(\pm\frac{\sqrt{2}}{3}\right)\left\{x - \left(4.\frac{2}{9} + 3\right)\right\}$$

or $\quad y \mp \frac{16\sqrt{2}}{27} + 1 \pm \sqrt{2}\left(x - \frac{35}{9}\right) \quad$ or $\quad 27(y + 1) \mp 16\sqrt{2} = \pm\sqrt{2}\,(27x - 105)$.

B. TANGENTS AT THE ORIGIN

If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.

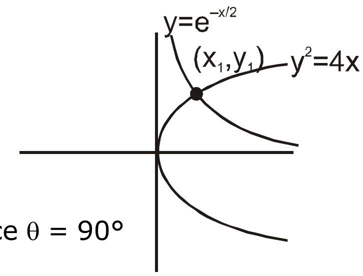
e.g., if the equation of a curve be $x^2 - 4y^2 + x^4 + 3x^3y + 3x^2y^2 + y^4 = 0$, the tangents at the origin are given by $x^2 - 4y^2 = 0$ or $x + 2y$ and $x - 2y = 0$.

In the curve $x^2 + y^2 + ax + by = 0$, $ax + by = 0$, is the equation of the tangent at the origin; and in the curve $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$, $x^2 - y^2 = 0$ is the equation of a pair of tangents at the origin.
If the equation of a curve be $x^2 + y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

C. ANGLE OF INTERSECTION

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection . If the angle between two curves is 90° then they are called **ORTHOGONAL** curves.

Ex.11 Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$



Sol. Let the curves intersect at point (x_1, y_1) for $y^2 = 4x$ $\quad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2}{y_1}$

and for $y = e^{-x/2}$ $\quad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{1}{2}e^{-x_1/2} = -\frac{y_1}{2}$ $\Rightarrow m_1, m_2 = -1$ Hence $\theta = 90°$

Note : here that we have not actually found the intersection point but geometrically we can see that the curves intersect.

Ex.12 Show that the curves $y = 2\sin^2 x$ and $y = \cos 2x$ intersect at $\pi/6$. What is their angle of intersection ?

Sol. Given curves are $y = 2\sin^2 x$ ...(1) and $y = \cos 2x$ ...(2)
Solving (1) and (2), we get $2\sin^2 x = \cos 2x$

$\Rightarrow \quad 1 - \cos 2x = \cos 2x \quad \Rightarrow \quad \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3} \quad \Rightarrow \quad 2x = \pm\frac{\pi}{3}$

$\therefore \quad x = \pm\frac{\pi}{6}$ are the points of intersection

From (1), $\frac{dy}{dx} = 4\sin x\cos x = 2\sin 2x = m_1$ (say) $\qquad$ From (2), $\frac{dy}{dx} = -2\sin 2x = m_2$ (say)

If angle of intersection is θ, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{4\sin 2x}{1 - 4\sin^2 2x}\right|$

$\therefore \quad (\tan\theta)_{x = \pm\pi/6} = \left|\frac{4 \times \pm\frac{\sqrt{3}}{2}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\pm 2\sqrt{3}}{-2}\right| = \sqrt{3} \qquad \therefore \quad \theta = \frac{\pi}{3}$

Ex.13 Show that the angle between the tangents at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\ln(x^2 + y^2) = c\tan^{-1}(y/x)$ where c is constant.

Sol. Let the point P(x, y) on the curve $\ln(x^2 + y^2) = c\tan^{-1}(y/x)$

Differentiating both sides w.r.t. x, we get $\frac{2x + 2yy'}{(x^2 + y^2)} = \frac{c(xy' - y)}{(x^2 + y^2)} \Rightarrow \quad y' = \frac{2x + cy}{cx - 2y} = m_1$ (say)

Slope of OP = $\frac{y}{x}$ = m_2 (say)

Let the angle between the tangents at P and OP be θ

$\therefore$ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2x+cy}{cx-2y} - \frac{y}{x}}{1 + \frac{2xy + cy^2}{cx^2 - 2xy}}\right| = \frac{2}{c}$. $\therefore$ $\theta = \tan^{-1}\left(\frac{2}{c}\right)$ which is independent of x and y.

Ex.14 Show tht the curves $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$ intersect orthogonally.

Sol. Given $\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1}$ = 1 ...(1) and $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2}$ = 1 ...(2)

Subtracting (2) from (1), we get $x^2\left(\frac{1}{a^2+k_1} - \frac{1}{a^2+k_2}\right) + y^2\left(\frac{1}{b^2+k_1} - \frac{1}{b^2+k_2}\right) = 0$

$\Rightarrow$ $x^2\left(\frac{k_2 - k_1}{(a^2+k_1)(a^2+k_2)}\right) + y^2\left(\frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)}\right) = 0$ $\therefore$ $\frac{x^2}{y^2} = -\frac{(a^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_1)}$ ...(3)

Now from (1), $\frac{2x}{(a^2+k_1)} + \frac{2y}{(b^2+k_1)}\frac{dy}{dx} = 0$ $\therefore$ $\frac{dy}{dx} = -\frac{x(b^2+k_1)}{y(a^2+k_1)}$ = m_1 (say)

Similarly from (2), $\frac{dy}{dx} = -\frac{x(b^2+k_2)}{y(a^2+k_2)}$ = m_2 (say) $\Rightarrow$ $m_1 m_2 = \frac{x^2(b^2+k_1)(b^2+k_2)}{y^2(a^2+k_1)(a^2+k_2)}$ = –1 [From (3)]

Hence given curves intersect orthogonally.

Ex.15 Prove that the curves xy = 4 and $x^2 + y^2 = 8$ touch each other.

Sol. Equation of the given curves are xy = 4 ....(i) and $x^2 + y^2 = 8$ ....(ii)

from (i), $1.y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ....(iii), from (ii), $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ ....(iv)

Putting the value fo y from (i) i (ii), we get $x^2 + \frac{16}{x^2} = 8$ or $x^4 + 16 = 8x^2$

or $x^4 - 8x^2 + 16 = 0$ or $(x^2 - 4)^2 = 0$ or $x^2 - 4 = 0$ or $x^2 = 4$ $\therefore$ $x = \pm 2$

from (i) ; when x = 2, y = $\frac{4}{2}$ = 2 and when x = –2, y = $\frac{4}{-2}$ = –2

Hence points of intersection of the two curves are (2, 2) and (–2, –2).

Slope of the tangent to the curve (i) at point (2, 2) $\Rightarrow m_1 = -\frac{2}{2} = -1$ [from (iii)]

Slope of tangent to the curve (ii) at point (2, 2) $\Rightarrow$ $m_2 = -\left(\frac{2}{2}\right) = -1$ [from (iv)]

$\because$ $m_1 = m_2$, therefore, the two curves have a common tangent at (2, 2) i.e. they touch each other at (2, 2).
At point (–2, –2) :

Slope of tangent to curve (i), $m_3 = -\left(\frac{-2}{-2}\right) = -1$

Slope of tangent to curve (ii), $m_4 = -\left(\frac{-2}{-2}\right) = -1$

Since $m_3 = m_4$, hence the two curves touch each other at (–2, –2). Thus curves (i) and (ii) touch each other.

Ex.16 Prove that the curves y = f(x) . f(x) > 0 and y = f(x) sin x, where f(x) is a differentiable function, have common tangent at common points.

Sol. The points of intersection of two curves are given by f(x) = f(x) sin x $\Rightarrow$ sin x = 1 $\Rightarrow$ x = (4n + 1)π/2
The slope of the tangent to y = f(x) at x = (4n + 1)π/2 is f′((4n + 1)π/2)
Slope of the tangent to y = f(x) sin x is f′(x) sin x + cos x f(x). So the slope of tangent at x = (4n + 1)π/2 is f′((4n + 1)π/2). 1 + 0 = f′((4n + 1)π/2). Here the slopes of the tangents are same at common points. Hence the two curves have the same tangent at their points of intersection.

Ex.17 The gradient of the common tangent to the two curves $y = x^2 - 5x + 6$ & $y = x^2 + x + 1$ is
(A) – 1/3 (B) – 2/3 (C) – 1 (D) – 3

Sol. y = ax + b on solving with both curves and putting D = 0 gives
$a^2 + 10a + 4b + 1 = 0$ and $a^2 - 2a + 4b - 3 = 0$ $\Rightarrow$ a = – 1/3 & b = 5/9
$\Rightarrow$ 3x + 9y = 5 ; point of contact (7/3, – 2/9) & (– 2/3, 7/9)

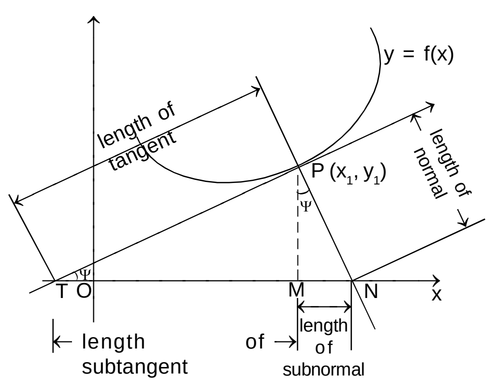
D. LENGTH OF TANGENT

(a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Subnormal (MN) = $y_1\, f'(x_1)$



Ex.18 What should be the value of n in the equation of curve $y = a^{1-n}.\, x^n$, so that the sub-normal may be of constant length ?

Sol. Given curve is $y = a^{1-n}\,.\,x^n$
Taking logarithm of both sides, we get, $\ln y = (1 - n) \ln a + n \ln x$

Differentiating both sides w.r.tx, we get $\frac{1}{y}.\frac{dy}{dx} = 0 + \frac{n}{x}$ or $\frac{dy}{dx} = \frac{ny}{x}$ ...(1)

$\therefore$ Lengths of sub-normal = $y\dfrac{dy}{dx} = y\,.\,\dfrac{ny}{x}$ {from (1)]

$= \dfrac{ny^2}{x} = n.\dfrac{(a^{1-n}x^n)^2}{x}$ $(\because\ y = a^{1-n}\,.\,x^n)$ $= n\,.\,a^{2-2n}\,.\,x^{2n-1}$

Since lengths of sub-normal is to be constant, so x should not appear in its value i.e., $2n - 1 = 0$.
$\therefore$ $n = 1/2$.

Ex.19 If the relation between sub-normal SN and sub-tangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$; then p/q is

(A) 8b/27 (B) b (C) 1 (D) none of these

Sol. $b \times 2y\dfrac{dy}{dx} = 3(x + a)^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{3}{2}\dfrac{(x+a)^2}{by} \Rightarrow \dfrac{p}{q} = \dfrac{(S_T)^2}{S_N} = \left|\dfrac{y_0}{(f'(x_0))^3}\right|$

Let a point by (x_0, y_0) lying on the curve $by_0^2 = (x_0 + a)^3$ ...(i)

$$\frac{p}{q} = \left|\frac{y_0}{\left(\frac{3}{2}\frac{(x_0+a)^2}{by_0}\right)^3}\right| = \left|\frac{8y_0^4 \times b^3}{27(x_0+a)^6}\right| = \frac{8}{27}b \qquad \text{(from equation (i))}$$

Ex.20 For the curve $y = a\,\ell n\,(x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Sol. Let point of tangency be $(x_1, y_1) \Rightarrow m = \left.\dfrac{dy}{dx}\right|_{x_1} = \dfrac{2ax_1}{x_1^2 - a^2}$

tangent + subtangent = $y_1\sqrt{1+\dfrac{1}{m^2}} + \dfrac{y_1}{m} = y_1\sqrt{1+\dfrac{(x_1^2-a^2)^2}{4a^2x_1^2}} + \dfrac{y_1(x_1^2-a^2)}{2ax_1}$

$= y_1\dfrac{\sqrt{x_1^4+a^4+2a^2x_1^2}}{2ax_1} + \dfrac{y_1(x_1^2-a^2)}{2ax_1} = \dfrac{y_1(x_1^2+a^2)}{2ax_1} + \dfrac{y_1(x_1^2-a^2)}{2ax_1} = \dfrac{y_1(x_1^2)}{2ax_1} = \dfrac{x_1y_1}{2a}$ Hence proved.

Ex.21 Show that the segment of the tangent to the curve $y = \dfrac{a}{2}\ell n\left(\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right) - \sqrt{(a^2-x^2)}$ contained between the y-axis and point of tangency has a constant length.

Sol. Let $x = a \sin\phi$ then $y = \frac{a}{2}\ell n\left(\frac{a + a\cos\phi}{a - a\cos\phi}\right) - a\cos\phi \Rightarrow y = a\ \ell n \cot\phi/2 - a\cos\phi$

$\therefore \frac{dx}{d\phi} = a\cos\phi$ and $\frac{dy}{d\phi} = \frac{a}{\sin\phi} + \sin\phi = \frac{a\cos^2\phi}{\sin\phi}$. Hence $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\phi}\right)}{\left(\frac{dx}{d\phi}\right)} = -\cot\phi$

Equation of tangent at 'ϕ' $\quad y - a\ \ell n \cot\phi/2 + a\cos\phi = \frac{-\cos\phi}{\sin\phi}(x - a\sin\phi)$

$\Rightarrow\ y\sin\phi - a\sin\phi\ \ell n \cot\phi/2 + a\sin\phi\cos\phi\ -x\cos\phi + a\sin\phi\cos\phi$

$\Rightarrow\ x\cos\phi + y\sin\phi = a\sin\phi\ \ell n \cot\phi/2$

Point on y-axis $P \equiv (0, a\ \ell n \cot\phi/2)$ and point of tangency $Q \equiv (a\sin\phi, a\ \ell n \cot\phi/2 - a\cos\phi)$

$\therefore\ PQ = \sqrt{(a^2\sin^2\phi + a^2\cos^2\phi)} = \sqrt{a^2} = a =$ constant.

E. SOLVING EQUATIONS

Ex.22 For what values of c does the equation $ln\ x = cx^2$ have exactly one solution ?

Sol. Let's start by graphing $y = ln\ x$ and $y = cx^2$ for various values of c. We know that for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect $y = ln\ x$ at all and one intersect twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

To find that particular value of c, we let 'a' be the x-coordinate of the single point of intersection. In other words, $ln\ a = ca^2$, so 'a' is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when x = a. That means the curves $y = ln$ x and $y = cx^2$ have the same slope when x = a. Therefore $\frac{1}{a} = 2ca$

Solving the equation $ln\ a = ca^2$ and $1/a = 2ca$, we get $ln\ a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$

Thus, $a = e^{1/2}$ and $\qquad c = \frac{ln\ a}{a^2} = \frac{ln\ e^{1/2}}{e} = \frac{1}{2e}$

For negative values of c we have the situation illustrated in Figure 3: All parabolas $y = cx^2$ with negative values of c intersect $y = ln\ x$ exactly once. And let's not forget about c = 0: The curve $y = 0\ x^2 = 0$ just he x-axis, which intersects $y = ln\ x$ exactly once.

To summarize, the required values of c are $c = 1/(2e)$ and $c \le 0$.

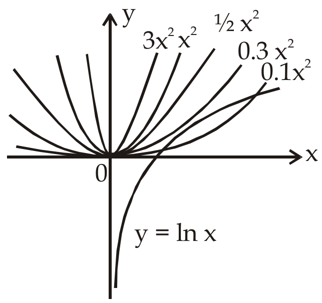


Figure 1

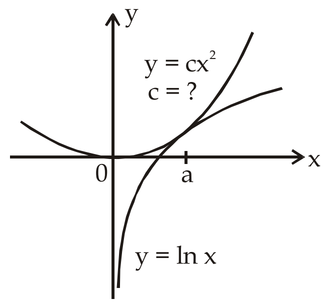


Figure 2

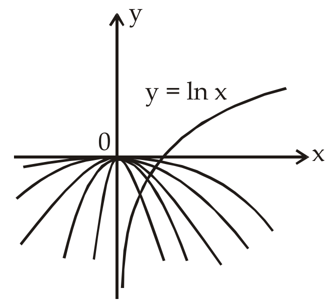


Figure 3

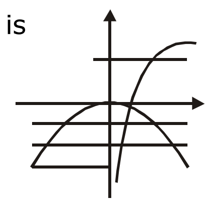
Ex.23 The set of values of p for which the equation $px^2 = \ln x$ possess a single root is

Sol. for $p \le 0$, there is obvious one solution ; for $p > 0$ one root

$\Rightarrow$ the curves touch each . $2px_1 = 1/x_1 \Rightarrow x_1^2 = 1/2p$;

Also $px_1^2 = \ln x_1 \Rightarrow p(1/2p) = \ln x_1 \Rightarrow x_1 = e^{1/2}$

$\Rightarrow 2p = 1/e \Rightarrow p = 1/2e$. Hence $p \in (-\infty, 0] \cup \{1/2e\}$



F. SHORTEST DISTANCE

Shortest distance between two non-intersecting curves always along the common normal (wherever defined)

Ex.24 Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$ then $\left.\dfrac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of line

$\Rightarrow 2x_1 + 3 = 1 \quad \Rightarrow \quad x_1 = -1 \quad \Rightarrow \quad y_1 = 0$

Hence point (–1, 0) is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance $\Rightarrow \; p = \dfrac{3}{\sqrt{2}}$.

Ex.25 Let P be a point on the curve C_1: $y = \sqrt{2 - x^2}$ and Q be a point on the curve C_2: $xy = 9$, both P and Q lie in the first quadrant. If 'd' denotes the minimum value between P and Q, find the value of d^2.

Sol. Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.

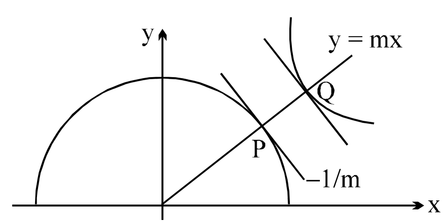
PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$

Let the normal at P be $y = mx$ ....(1) $(m > 0)$ solving it with $xy = 9$

$mx^2 = 9 \Rightarrow x = \dfrac{3}{\sqrt{m}}$; $y = \dfrac{9\sqrt{m}}{3} \quad \therefore \quad Q \equiv \left(\dfrac{3}{\sqrt{m}}, 3\sqrt{m}\right)$

differentiating $xy = 9$

$$x\frac{dy}{dx} + y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$



$$\therefore \left.\frac{dy}{dx}\right|_Q = -\frac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m \qquad \therefore \text{ tangent at P and Q must be parallel}$$

$$\therefore -m = -\frac{1}{m} \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = 1 \quad \therefore \text{ normal at P and Q is } y = x$$

solving P(1, 1) and Q(3, 3) $\therefore \; (PQ)^2 = d^2 = 4 + 4 = 8$

G. RATE MEASUREMENT

Ex.26 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

Sol. We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both function of t (time). We are given that dx/dt = 1 ft/s and we are asked to find dy/dt when x = 6 ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem : $x^2 + y^2 = 100$

Differentiating each side with respect to t using the Chain Rule, we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

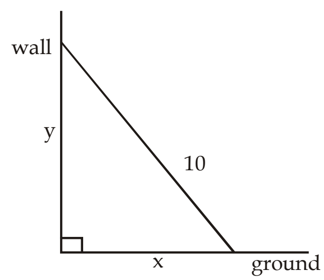
and solving this equation for the desired rate, we obtain $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When x = 6, the Pythagorean Theorem gives y = 8

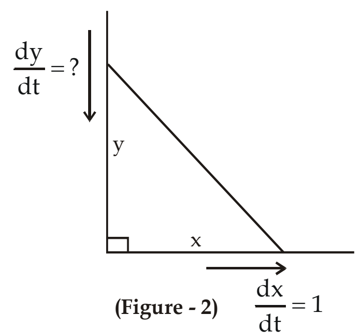
and so, substituting these values and dx/dt = 1,

we have $\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}$ ft/s

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is decreasing at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s.



(Figure - 1)



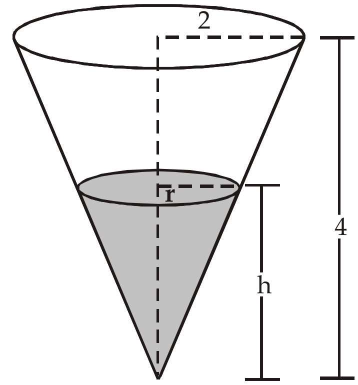
(Figure - 2)

Ex.27 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Sol. We first sketch the cone and label it as in Figure. Let V, r, and h be the volume of the water, the radius of the surface, and the height at time t, where t is measured in minutes.

We are given that dV/dt = 2m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation V = $\frac{1}{3}\pi r^2$ h. But it is the very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure to write $\frac{r}{h} = \frac{2}{4}$ r = $\frac{h}{2}$

and the expression for V becomes V = $\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$



Figure

Now we can differentiate each side with respect to t :

$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$

Substituting h = 3 m and dV/dt = 2m^3/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min.

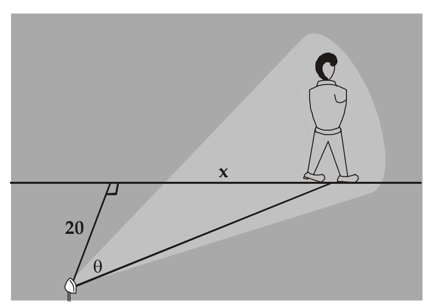
Ex.28 A man walks along a straight path at the speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight ?

Sol. We draw Figure and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the distance from the man to the point on the path closest to the searchlight and the perpendicular to the path.

We are given that dx/dt = 4 ft/s and are asked to find dθ/dt when x = 15. The equation that relates x and θ can be written from Figure. $\frac{x}{20}$ = tan θ x = 20 tan θ

Differentiating each side with respect to t, we get $\frac{dx}{dt}$ = 20 sec$^2\theta$ $\frac{d\theta}{dt}$

so $\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\frac{dx}{dt} = \frac{1}{20}\cos^2\theta(4) = \frac{1}{5}\cos^2\theta$



when x = 15, the length of the beam is 25, so cos $\theta = \frac{4}{5}$

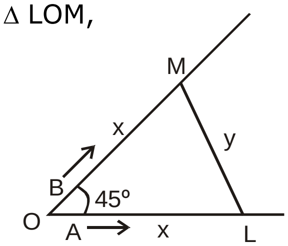
and $\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125}$ = 0.128

The searchlight is rotating at a rate of 0.128 rad/s.

Ex.29 Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45º to each other. If they travel by different roads, find the rate at which they are being separated.

Sol. Let L and M be the positions of men A and B at any time t,

Let OL = x and LM = y. Then OM = x given, $\frac{dx}{dt}$ = v; to find $\frac{dy}{dt}$ from Δ LOM,



$$\cos 45^\circ = \frac{OL^2 + OM^2 - LM^2}{2 . OL . OM} \text{ or, } \frac{1}{\sqrt{2}} = \frac{x^2 + x^2 - y^2}{2 . x . x} = \frac{2x^2 - y^2}{2x^2}$$

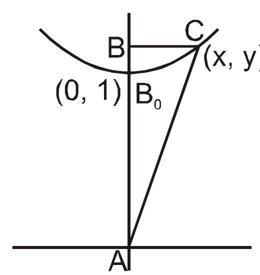
or, $\sqrt{2x^2}$ = 2x² – y² or $(2 - \sqrt{2})\, x^2 = y^2$ $\therefore$ $y = \frac{dy}{dt} = \sqrt{2-\sqrt{2}}\, x$

differentiating w. r. t. we get $\frac{dy}{dt} = \sqrt{2\sqrt{2}}\,\frac{dx}{dt} = \sqrt{2-\sqrt{2}}\, v$ $\left[\because \frac{dx}{dt} = v\right]$

$\therefore$ they are being separated from each other at the rate $\sqrt{2-\sqrt{2}}$ v.

Ex.30 A variable triangle ABC in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin & the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 & moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Sol. $A = \frac{xy}{2} = \frac{x}{2}\left(1 + \frac{7x^2}{36}\right)$; $\frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24}x^2\right)\frac{dx}{dt}$



at $t = \frac{7}{2}$; $y = 2 \times \frac{7}{2} = 7$ $\Rightarrow$ AB = 8

when y = 8 then x = 6 $\Rightarrow \frac{dA}{dt} = \left(\frac{1}{2} + \frac{7}{24} 3.6\right)\frac{dx}{dt} = 11 . \frac{dx}{dt}$

Also $\frac{dy}{dt} = 2 = \frac{14x}{36}\frac{dx}{dt}$ $\Rightarrow$ $\frac{dx}{dt} = \frac{36}{7x} = \frac{6}{7}$ $\Rightarrow$ $\frac{dA}{dt} = 11 . \frac{6}{7} = \frac{66}{7}$

Ex.31 Find the approximate value of $(1.999)^6$.

Sol. Let $f(x) = x^6$. Now, $f(x + \delta x) - f(x) = f'(x) . \delta x = 6x^5\, \delta x$

We may write, 1.999 = 2 – 0.001

Taking x = 2 and $\delta x = -0.001$, we have $f(1.999) - f(2) = 6(2)^5 \times -0.001$

$\Rightarrow$ f(1.999) = f(2) – 6 × 32 × 0.001 = 64 – 64 × 0.003 = 64 × 0.997 = 63.808 (approx).